Examen 1 (Solutions)
201-GNF Calcul 3
♡ 14 février 2019 ♡
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Question 1. (10%)

Associer chacunes des courbes suivantes avec la fonction vectorielle qui lui correspond

A. IV B. II C. VI D. I E. V F. III

Question 2. (10%)

On peut utiliser la première composante pour trouver le t. $1 = 2t + 5 \Longrightarrow t = -2$. $\vec{r}\,'(t) = (2, 2t, 3t^2)$ d'où $\vec{r}\,'(-2) = (2, -4, 12)$. Finalement, l'équation de la droite tangente est

$$(x, y, z) = (1, 0, -1) + k(2, -4, 12)$$

Question 3. (10%)

On a
$$\vec{r}(t) = e^t(\cos t, \sin t) \quad \text{et} \quad \vec{r}\,'(t) = e^t(\cos t, \sin t) + e^t(-\sin t, \cos t)$$
et
$$\vec{r}(t) \cdot \vec{r}\,'(t) = e^{2t}(\cos^2 t + \sin^2 t) + 0 = e^{2t}$$
$$\|\vec{r}(t)\| = e^t\sqrt{\cos^2 t + e\sin^2 t} = e^t$$
$$\|\vec{r}(t)'\| = e^t\sqrt{(\cos^2 t - 2\cos t\sin t + \sin^2 t) + (\cos^2 t + 2\cos t\sin t + \sin^2 t)} = \sqrt{2}e^t$$
donc
$$\cos\theta = \frac{\vec{r}(t) \cdot \vec{r}\,'(t)}{\|\vec{r}(t)\|\|\vec{r}\,'(t)\|} = \frac{e^{2t}}{e^t\sqrt{2}e^t} = \frac{\sqrt{2}}{2} \Longrightarrow \theta = \frac{\pi}{4}$$

Question 4. (10%)

$$\vec{r}\,'(t) = (1, \tan t, \sec t) \quad \text{et} \quad \|\vec{r}\,'(t)\| = \sqrt{1 + \tan^2 t + \sec^2 t} = \sqrt{2\sec^2 t} = \sqrt{2}|\sec t|$$
$$L = \sqrt{2}\int_0^{\frac{\pi}{4}} \sec t\ dt = \sqrt{2}\ln|\sec t + \tan t|\Big|_0^{\frac{\pi}{4}} = \sqrt{2}(\ln(\sqrt{2}+1) - \ln(1+0)) = \sqrt{2}\ln(\sqrt{2}+1)$$

Question 5. (20%)

a) $\vec{r}\,'(t) = (1, t, -1)$ et $\|\vec{r}\,'(t)\| = \sqrt{2 + t^2}$ donc $\vec{T}(t) = \left(\frac{1}{\sqrt{2+t^2}}, \frac{t}{\sqrt{2+t^2}}, \frac{-1}{\sqrt{2+t^2}}\right)$

$$\vec{T}(1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

b) $\vec{T}'(t) = \left(\dfrac{-t}{\sqrt{(2+t^2)^3}}, \dfrac{2+t^2-t^2}{\sqrt{(2+t^2)^3}}, \dfrac{t}{\sqrt{(2+t^2)^3}}\right) = \left(\dfrac{-t}{\sqrt{(2+t^2)^3}}, \dfrac{2}{\sqrt{(2+t^2)^3}}, \dfrac{t}{\sqrt{(2+t^2)^3}}\right)$

$\vec{T}'(1) = \left(\dfrac{-1}{\sqrt{3^3}}, \dfrac{2}{\sqrt{3^3}}, \dfrac{1}{\sqrt{3^3}}\right)$, $\|\vec{T}'(1)\| = \sqrt{\dfrac{1^2}{3^3}+\dfrac{2^2}{3^3}+\dfrac{1^2}{3^3}} = \sqrt{\dfrac{6}{3^3}}$ et donc

$$\vec{N}(1) = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

c)

$$\vec{B}(1) = \frac{1}{\sqrt{18}}\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} = \frac{1}{\sqrt{18}}(3,0,3) = \frac{1}{\sqrt{2}}(1,0,1)$$

d) $\left(x-1, y-\dfrac{1}{2}, z-1\right)) \cdot (1,0,1) = 0 \Longrightarrow x+z=2$

Question 6. (15%)

Soit la courbe $\vec{r}(t) = (t^2, t^3+t, 2-t)$, trouver (sans simplifier)

$$\vec{r}\,'(t) = (2t, 3t^2+1, -1), \quad \vec{r}\,''(t) = (2, 6t, 0), \quad \vec{r}\,'''(t) = (0,6,0), \quad \|\vec{r}\,'(t)\| = \sqrt{9t^4+10t^2+2}$$

$$\vec{r}\,'(t) \wedge \vec{r}\,''(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t & 3t^2+1 & -1 \\ 2 & 6t & 0 \end{vmatrix} = (6t, -2, 6t^2-2)$$

a) $\kappa(t) = \dfrac{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \sqrt{\dfrac{36t^2+4+(6t^2-2)^2}{(9t^4+10t^2+2)^3}} = \sqrt{\dfrac{36t^4+12t^2+8}{(9t^4+10t^2+2)^3}}$

b) $\rho(t) = \dfrac{1}{\kappa(t)} = \sqrt{\dfrac{(9t^4+10t^2+2)^3}{36t^4+12t^2+8}}$

c) $\tau(t) = \dfrac{(\vec{r}\,'(t)\wedge\vec{r}\,''(t))\cdot\vec{r}\,'''(t)}{\|\vec{r}\,'(t)\wedge\vec{r}\,''(t)\|^2} = \dfrac{-12}{36t^4+12t^2+8}$

Question 7. (20%)

Une particule décrit une trajectoire dont l'accélération est donnée par $\vec{a}(t) = (e^t, 15\sqrt{t}, 12t^2)$. Si sa position initiale est $\vec{r}(0) = (2,1,3)$ et sa vitesse initiale est $\vec{v}(0) = (1,2,3)$, trouver

a) $\vec{v}(t) = \displaystyle\int \vec{a}(t)\,dt = \left(e^t + C_1, 10t^{\frac{3}{2}} + C_2, 4t^3 + C_3\right)$, $\vec{v}(0) = (1+C_1, C_2, C_3) = (1,2,3)$

$$\vec{v}(t) = \left(e^t, 10t^{\frac{3}{2}}+2, 4t^3+3\right)$$

b) $\vec{r}(t) = \displaystyle\int \vec{v}(t)\,dt = \left(e^t + C_4, 4t^{\frac{5}{2}} + 2t + C_5, t^4 + 3t + C_6\right)$, $\vec{r}(0) = (1+C_4, C_5, C_6) = (2,1,3)$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(e^t+1, 4t^{\frac{5}{2}}+2t+1, t^4+3t+3\right)$$

c) $\vec{a}(1) = (e, 15, 12)$, $\vec{v}(1) = (e, 12, 7)$,

$$a_T(t) = \frac{\vec{a}(t)\cdot\vec{v}(t)}{v(t)} = \frac{e^2+180+84}{\sqrt{e^2+144+49}} = \frac{e^2+264}{\sqrt{e^2+193}}$$

d) $\vec{a}(1) \wedge \vec{v}(1) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ e & 15 & 12 \\ e & 12 & 7 \end{vmatrix} = (-39, 5e, -3e)$

$$a_N(t) = \frac{\|\vec{a}(t)\wedge\vec{v}(t)\|}{v(t)} = \sqrt{\frac{39^2+25e^2+9e^2}{e^2+144+49}} = \sqrt{\frac{39^2+34e^2}{e^2+193}}$$

Question 8. (5%)

La courbe est dans le plan $z = 1$ donc elle a une torsion nulle.