Examen 4 (solutions)
201-GNF Calcul 3
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Question 1. (12%)

A. $\vec{F}(x,y) = (y, \sin x) \to \text{IV}$
C. $\vec{F}(x,y) = (x, 1) \to \text{II}$
E. $\vec{F}(x,y) = (x, y) \to \text{I}$
B. $\vec{F}(x,y) = (-y, x) \to \text{V}$
D. $\vec{F}(x,y) = (-x, y) \to \text{VI}$
F. $\vec{F}(x,y) = (\sin x, \sin y) \to \text{III}$

Question 2. (12%)

a)

$$\begin{aligned}\int_{\mathcal{C}} x^2 y\, ds &= \int_0^{\frac{\pi}{2}} (3\cos t)^2 (3\sin t) \sqrt{9\sin^2 t + 9\cos^2 t}\ dt \\ &= 3^4 \int_0^{\frac{\pi}{2}} \cos^2 t \sin t\ dt = -3^4 \frac{\cos^3 t}{3}\bigg|_0^{\frac{\pi}{2}} = 3^3\end{aligned}$$

b) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ où $\mathcal{C}_1 : \vec{r}_1(t) = (1+t, -t)$ et $\mathcal{C}_2 : \vec{r}_2(t) = (2-2t, 1+2t)$

$$\begin{aligned}\int_{\mathcal{C}} 2x - 4y\, dy &= \int_{\mathcal{C}_1} 2x - 4y\, dy + \int_{\mathcal{C}_2} 2x - 4y\, dy \\ &= \int_0^1 \Big(2(1+t) - 4(-t)\Big)(-1)\ dt + \int_0^1 \Big(2(2+2t) - 4(1+2t)\Big)(2)\ dt \\ &= \int_0^1 -2 - 6t\ dt + \int_0^1 -8t\ dt = -2t - \frac{14t^2}{2}\bigg|_0^1 = -9\end{aligned}$$

Question 3. (10%)

a) Ici $\mathcal{C}_1 : \vec{r}_1(t) = (1-t, -t)$ et donc

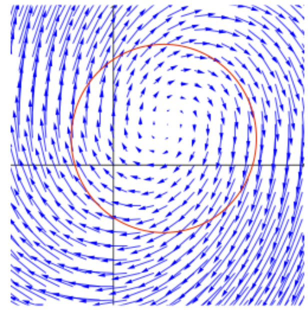
$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_0^1 (-t, t-1)\cdot(-1,-1)\ dt \\ &= \int_0^1 t - t + 1 dt = \int_0^1 dt = 1\end{aligned}$$

b) Ici $\mathcal{C}_1 : \vec{r}_1(t) = (\cos t, \sin t)$ et donc

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_0^{\frac{3\pi}{2}} (\sin t, -\cos t)\cdot(-\sin t, \cos t)\ dt \\ &= \int_0^{\frac{3\pi}{2}} -\sin^2 t - \cos^2 t dt = \int_0^{\frac{3\pi}{2}} -1\ dt = -\frac{3\pi}{2}\end{aligned}$$

Question 4. (10%)

On peut voir que $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} \neq 0$ car en tout point de la courbe en rouge, parcouru dans le sens des aiguilles d'une montre, le produit scalaire est positif donc $\vec{F}$ n'est pas conservatif.



Question 5. (14%)

a) $\vec{F}(x,y) = (6xy-4, 4y+3x^2)$ et $\dfrac{\partial Q}{\partial x} = 6x = \dfrac{\partial P}{\partial y}$

$$\int 6xy - 4\ dx = 3x^2y - 4x + g(y)$$
$$\int 4y + 3x^2\ dy = 2y^2 + 3x^2y + h(x)$$

D'où $f(x,y) = 3x^3y + 2y^2 - 4x + C$ est une fonction potentielle.

b) $\vec{F}(x,y,z) = \left(z^2ye^{xyz}, 2y + xz^2e^{xyz}, (1+xyz)e^{xyz}\right)$

$$\frac{\partial P}{\partial y} = z^2e^{xyz} + z^3yxe^{xyz} = \frac{\partial Q}{\partial x},$$
$$\frac{\partial P}{\partial z} = 2zye^{xyz} + z^2y^2xe^{xyz} = \frac{\partial R}{\partial x},$$
$$\frac{\partial Q}{\partial z} = 2zxe^{xyz} + z^2x^2ye^{xyz} = \frac{\partial R}{\partial y}.$$

$$\int z^2ye^{xyz}\ dx = ze^{xyz} + g(y,z)$$
$$\int 2y + xz^2e^{xyz}\ dy = y^2 + ze^{xyz} + h(x,z)$$
$$\int (1+xyz)e^{xyz}\ dz = ze^{xyz} + k(x,y)$$

D'où $f(x,y,z) = y^2 + ze^{xyz} + C$ est une fonction potentielle.

Question 6. (10%)

$$\begin{aligned}\int_0^1\int_{x^2}^x \frac{\partial}{\partial x}(x^2e^y)-\frac{\partial}{\partial y}(y\sin x)\ dy\ dx &= \int_0^1\int_{x^2}^x 2xe^y-\sin x\ dy\ dx\\ &= \int_0^1 2xe^y - y\sin x\Big|_{x^2}^{x}\ dx\\ &= \int_0^1 2xe^{x^2}-2xe^x-x^2\sin x+x\sin x\ dx\\ &= e^{x^2}-2xe^x+2e^x+(x^2-x-2)\cos x+(1-2x)\sin x\Big|_0^1\\ &= -1+e-\sin(1)-2\cos(1)\end{aligned}$$

Question 7. (10%)

$\vec{r}(t)=(\cos t\cos(4t),\sin t\cos(4t))$

$$\begin{aligned}\frac{1}{2}\oint_{\mathcal{C}} -y\,dx+x\,dy &= \frac{1}{2}\int_0^{2\pi} -y\,x'+x\,y'\,dt\\ &= \frac{1}{2}\int_0^{2\pi} -\sin t\cos(4t))\,x'+\cos t\cos(4t)\,y'\,dt\\ &= \frac{1}{2}\int_0^{2\pi} -\sin t\cos(4t))(-\sin t\cos 4t-4\cos t\sin 4t)+\cos t\cos(4t)(\cos t\cos 4t-4\sin t\sin 4t)\,dt\\ &= \frac{1}{2}\int_0^{2\pi} \sin^2 t\cos^2(4t)+4\sin t\cos t\cos 4t\sin 4t+\cos^2 t\cos^2(4t)-4\cos t\sin t\cos 4t\sin 4t\,dt\\ &= \frac{1}{2}\int_0^{2\pi}\cos^2(4t)\,dt=\frac{1}{2}\int_0^{2\pi}\frac{1+\cos(8t)}{2}\,dt=\frac{1}{4}\left(t+\frac{\sin(8t)}{8}\right)\Big|_0^{2\pi}=\frac{\pi}{2}\end{aligned}$$

Question 8. (12%)

a) Faux $\dfrac{\partial R}{\partial y}=\dfrac{\partial Q}{\partial z}$ mais en général $\dfrac{\partial R}{\partial y}\neq\dfrac{\partial P}{\partial z}$

b) Vrai car

$$\begin{aligned}\oint_{\mathcal{C}}(xy^2+2y)\,dx+(3x+x^2y)\,dy &= \iint_R \frac{\partial}{\partial x}(3x+x^2y)-\frac{\partial}{\partial y}(xy^2+2y)\ dA\\ &= \iint_R 3+2xy-(2xy+2)\ dA=\iint_R dA\end{aligned}$$

Question 9. (10%)

Puisque

$$\frac{\partial}{\partial x}\left(\frac{4y}{3x^2+2y^2}\right)=\frac{-24xy}{(3x^2+2y^2)^2}=\frac{\partial}{\partial y}\left(\frac{6x}{3x^2+2y^2}\right)$$

et que $\vec{F}$ est continue sur toute la région de b) le théorème de Green nous dit que l'intégrale en b) est 0é