

2.3 SUBSTITUTION TRIGONOMETRIQUE

cours 11

Au dernier cours, nous avons vu

✓ Identités trigonométriques

$$\sin^2 x + \cos^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x \quad \cot^2 x + 1 = \csc^2 x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

✓ Intégrale de la forme

$$\int \sin^n x \cos^m x \, dx$$

$$\int \tan^n x \sec^m x \, dx$$

Au dernier cours, nous avons vu

✓ Formule de réduction

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Aujourd'hui, nous allons voir

- ✓ Substitution trigonométrique

On sait intégrer les fonctions de la forme

$$f(x) = \frac{1}{ax + b} \qquad g(x) = \sqrt{ax + b}$$

Car les changements de variable linéaire

$$u = ax + b \qquad du = a \, dx$$

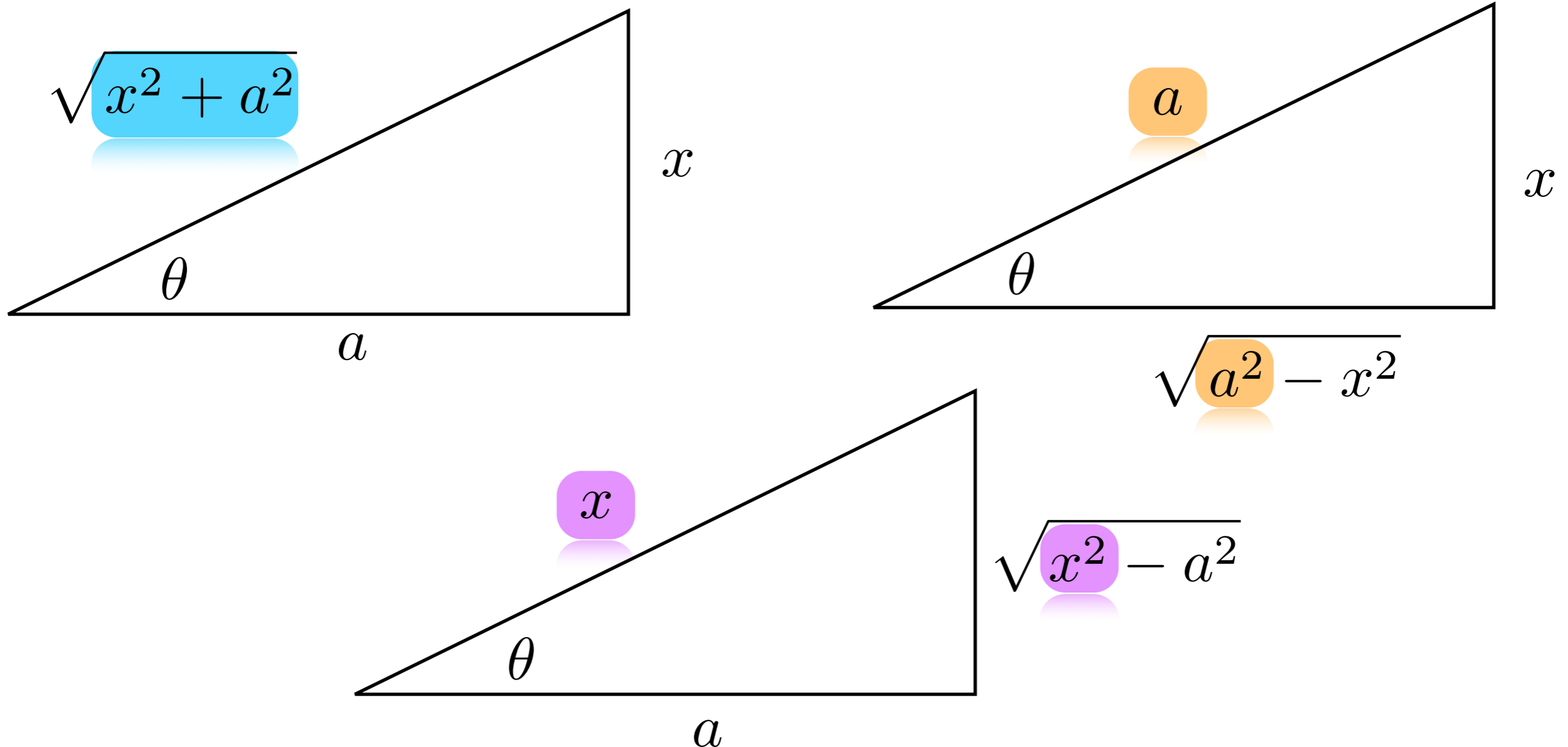
fonctionnent toujours.

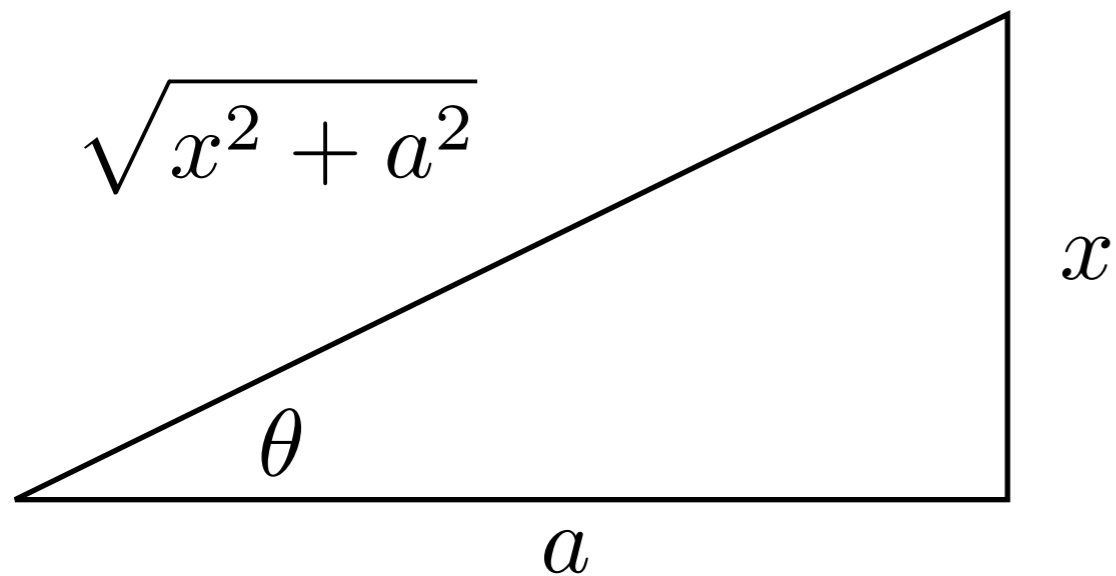
Aujourd'hui et dans les cours suivants, on va voir comment intégrer certaines fonctions composées dont l'intérieur est de la forme

$$ax^2 + bx + c$$

La substitution trigonométrique repose sur le théorème de Pythagore ainsi que les rapports trigonométriques.

Considérons une variable x , une constante a et écrivons l'autre côté d'un triangle rectangle en fonction de ces données



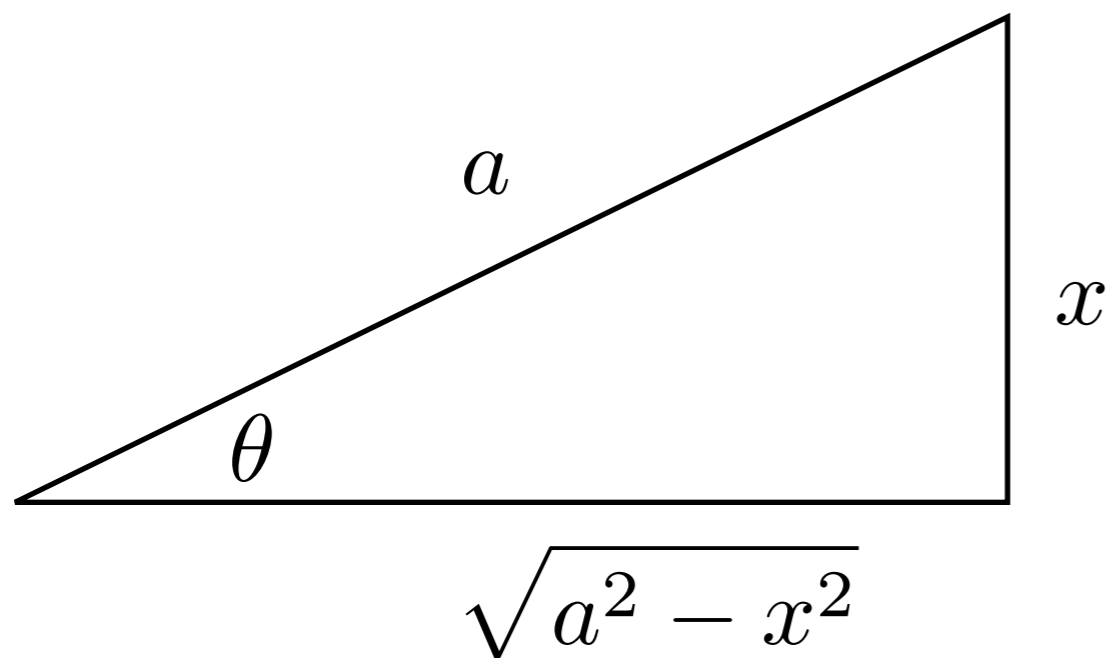


$$\frac{x}{a} = \tan \theta$$

$$x = a \tan \theta$$

$$\frac{\sqrt{x^2 + a^2}}{a} = \sec \theta$$

$$\sqrt{x^2 + a^2} = a \sec \theta$$

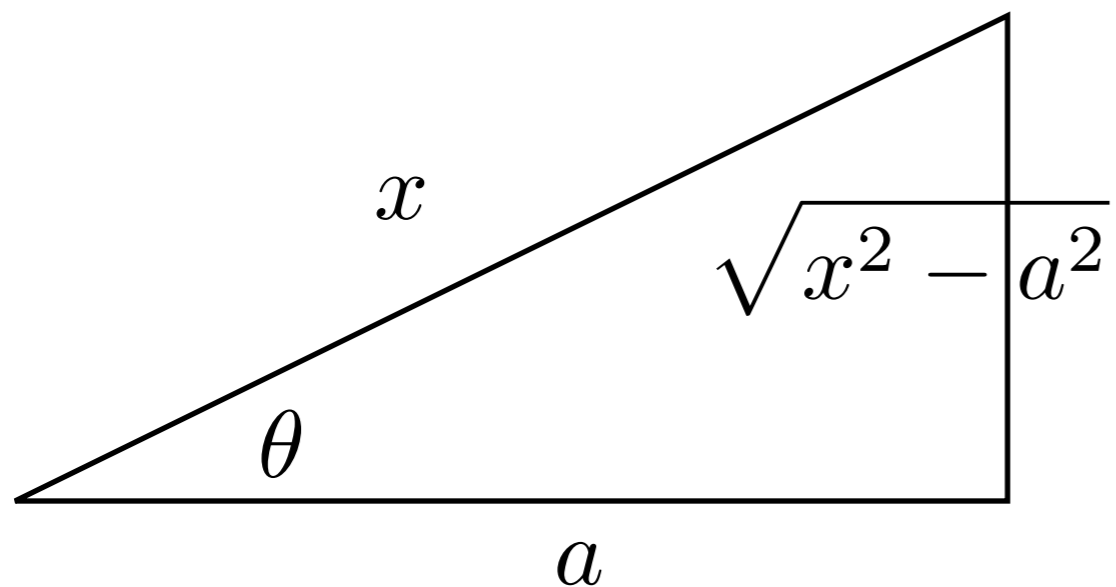


$$\frac{x}{a} = \sin \theta$$

$$x = a \sin \theta$$

$$\frac{\sqrt{a^2 - x^2}}{a} = \cos \theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta$$



$$\frac{x}{a} = \sec \theta$$

$$x = a \sec \theta$$

$$\frac{\sqrt{x^2 - a^2}}{a} = \tan \theta$$

$$\sqrt{x^2 - a^2} = a \tan \theta$$

Faites les exercices suivants

Section 2, # 11

Voyons voir comment mélanger ça avec des intégrales.

Exemple

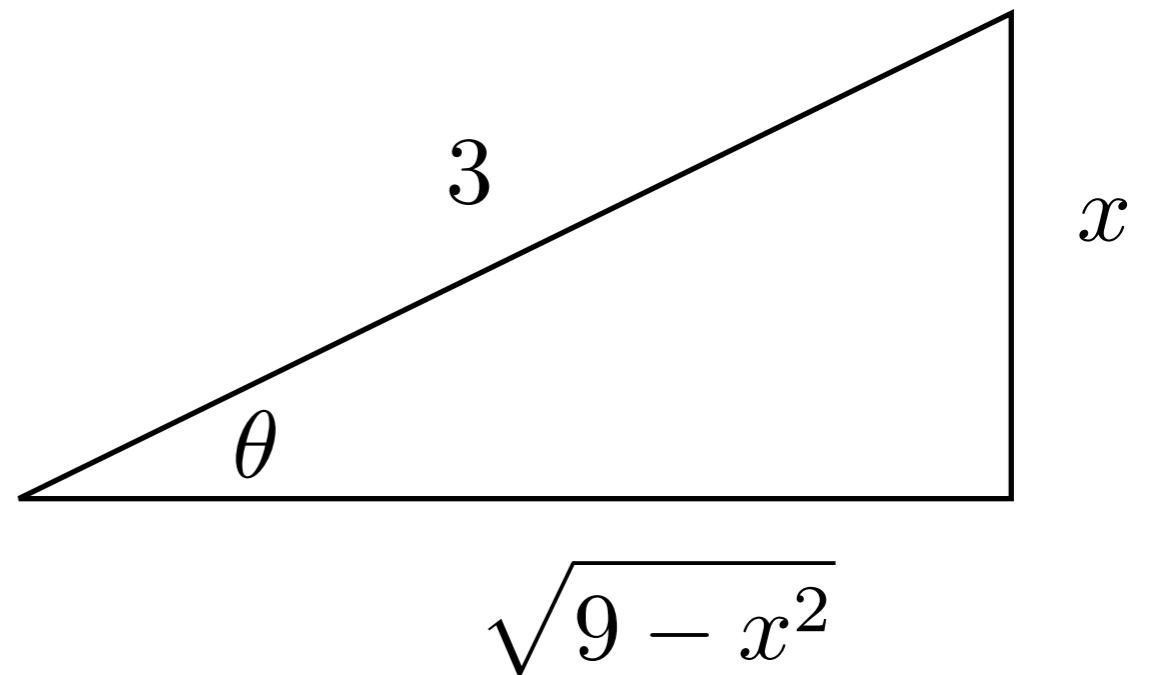
$$\int \frac{x}{\sqrt{9-x^2}} dx$$

$$= \int \frac{3 \sin \theta}{3 \cos \theta} 3 \cos \theta d\theta$$

$$= 3 \int \sin \theta d\theta$$

$$= -3 \cos \theta + C$$

$$= -3 \frac{\sqrt{9-x^2}}{3} + C$$



Notre changement de variable

$$\frac{x}{3} = \sin \theta \quad x = 3 \sin \theta$$

$$dx = 3 \cos \theta d\theta$$

$$\frac{\sqrt{9-x^2}}{3} = \cos \theta$$

$$\sqrt{9-x^2} = 3 \cos \theta$$

Faites les exercices suivants

Section 2, # 12 et 13

Exemple

$$\int \sqrt{4 - x^2} dx$$

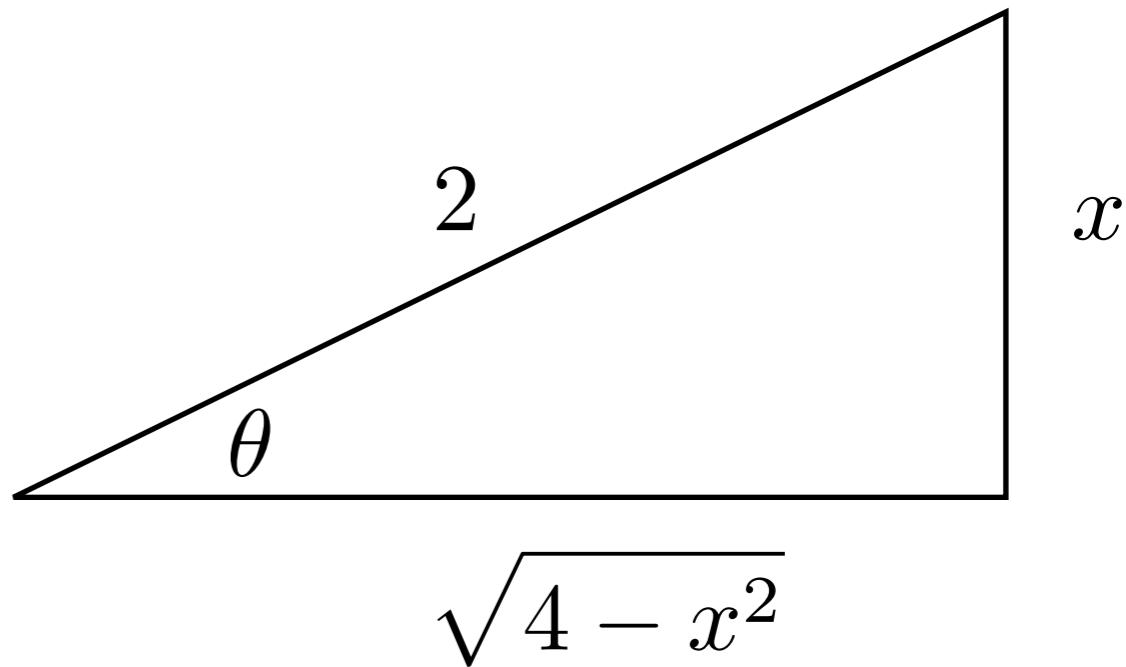
$$= \int (2 \cos \theta) 2 \cos \theta d\theta$$

$$= 4 \int \cos^2 \theta d\theta$$

$$= 4 \int \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= 2\theta + \sin(2\theta) + C$$

$$= 2 \arcsin \left(\frac{x}{2} \right) + 2 \sin \theta \cos \theta + C$$



$$\frac{x}{2} = \sin \theta$$

$$x = 2 \sin \theta$$

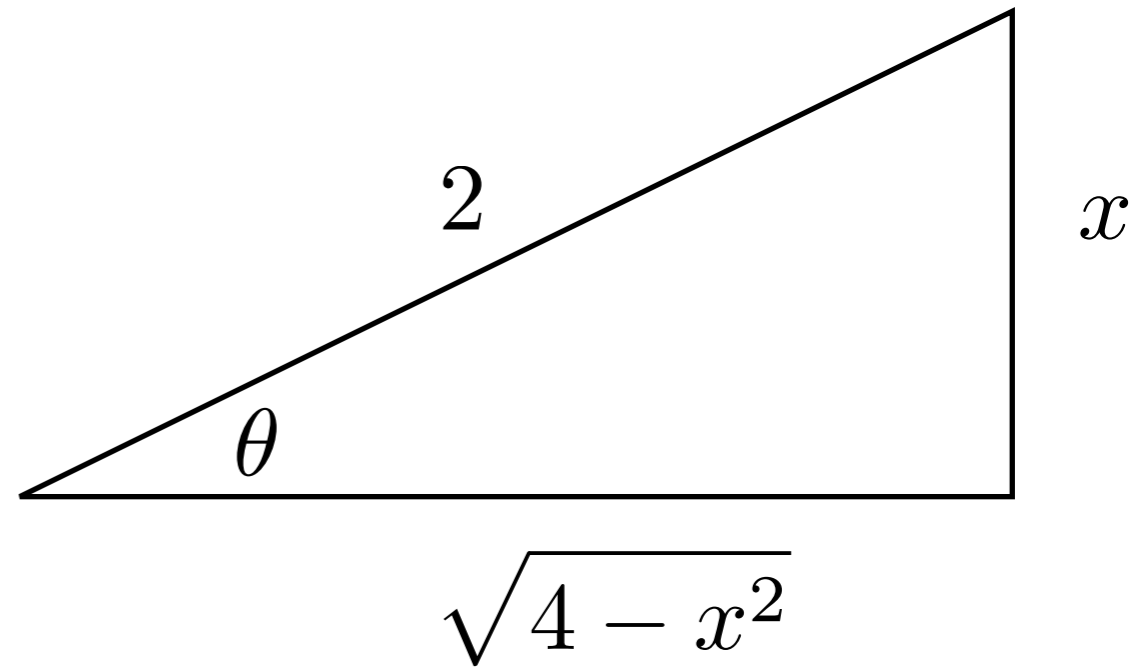
$$dx = 2 \cos \theta d\theta$$

$$\frac{\sqrt{4 - x^2}}{2} = \cos \theta$$

$$\sqrt{4 - x^2} = 2 \cos \theta$$

Example

$$\int \sqrt{4 - x^2} \, dx$$



$$= 2 \arcsin\left(\frac{x}{2}\right) + 2 \sin \theta \cos \theta + C$$

$$\frac{x}{2} = \sin \theta \quad x = 2 \sin \theta$$

$$= 2 \arcsin\left(\frac{x}{2}\right) + 2 \left(\frac{x}{2}\right) \left(\frac{\sqrt{4 - x^2}}{2}\right) + C$$

$$= 2 \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4 - x^2}}{2} + C$$

$$\frac{\sqrt{4 - x^2}}{2} = \cos \theta$$

Faites les exercices suivants

Section 2 , # 12, 13

Exemple

$$x^2 + y^2 = r^2$$

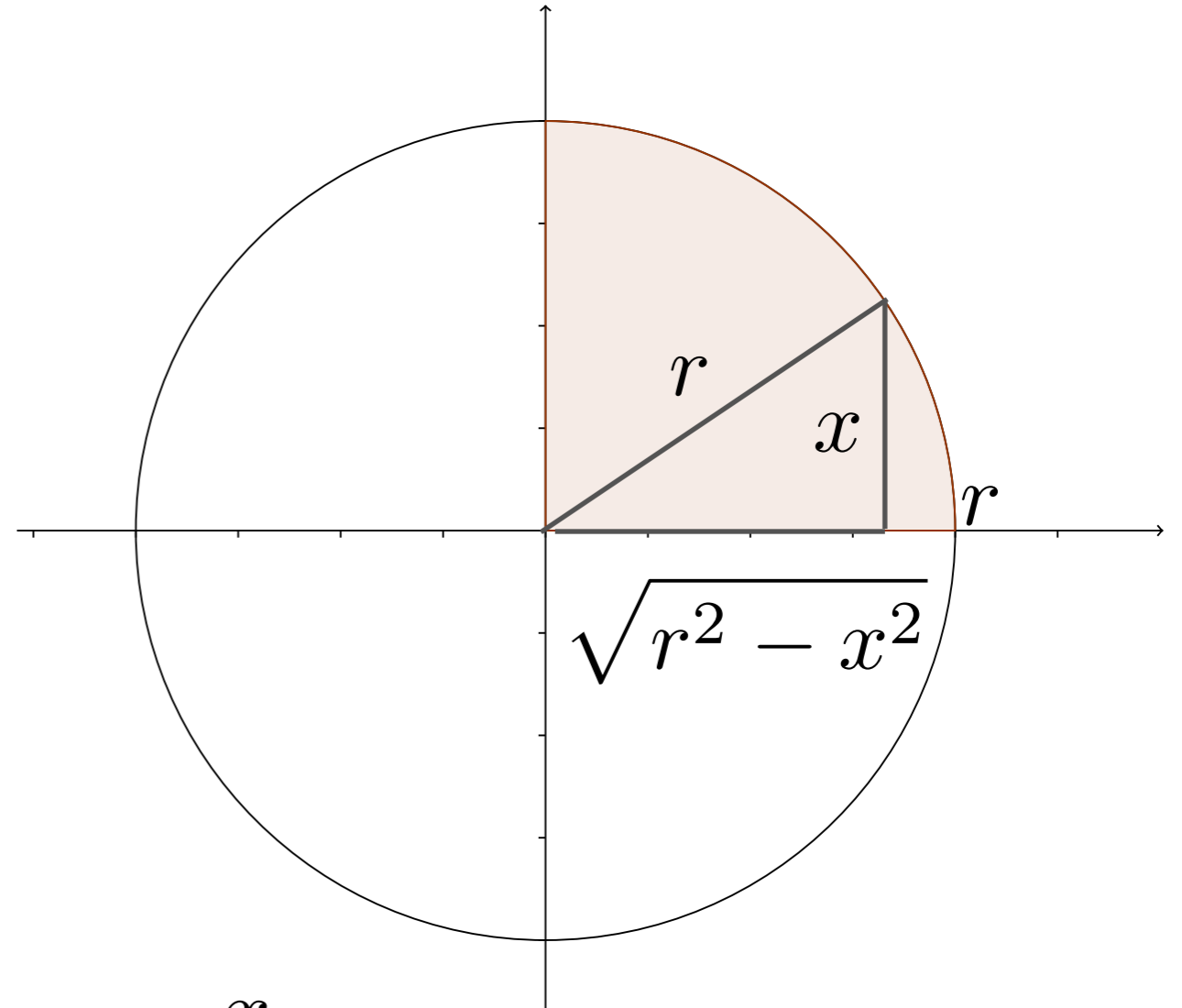
$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

$$\text{Aire}_o = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$= 4 \int_{?}^{?} r^2 \cos^2 \theta d\theta$$

$$= 4r^2 \int_{?}^{?} \frac{1 + \cos(2\theta)}{2} d\theta$$



$$\frac{x}{r} = \sin \theta \quad x = r \sin \theta$$

$$dx = r \cos \theta d\theta$$

$$\frac{\sqrt{r^2 - x^2}}{r} = \cos \theta$$

$$\sqrt{r^2 - x^2} = r \cos \theta$$

Exemple

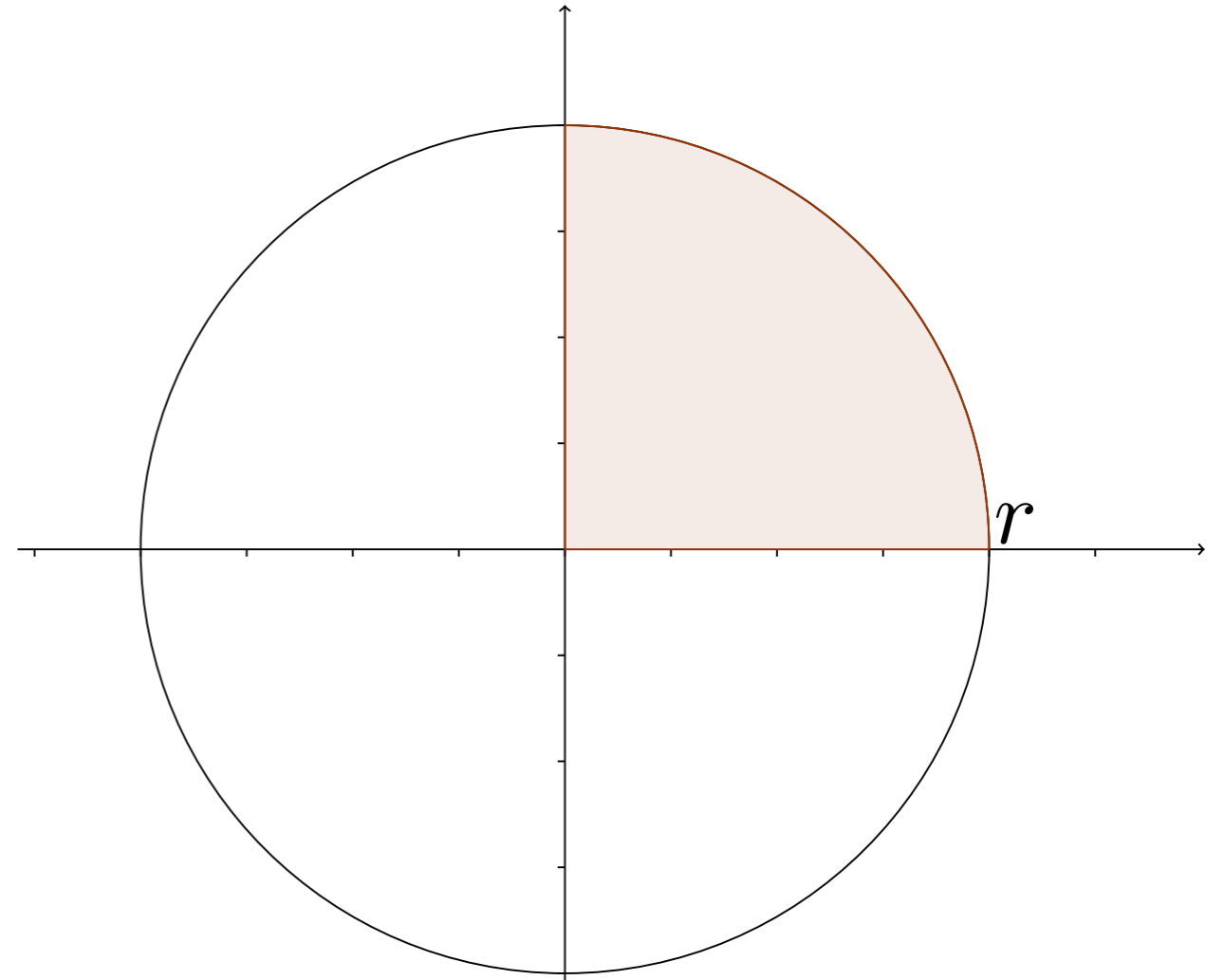
$$\text{Aire}_o = 4 \int_0^r \sqrt{r^2 - x^2} \, dx$$

$$= 4r^2 \int_{?}^{?} \frac{1 + \cos(2\theta)}{2} \, d\theta$$

$$= 2r^2 \int_{?}^{?} 1 + \cos(2\theta) \, d\theta$$

$$= 2r^2 \left(\theta + \frac{\sin(2\theta)}{2} \right) \Big|_{?}^{?}$$

$$= 2r^2 \left(\arcsin \left(\frac{x}{r} \right) + \sin \theta \cos \theta \right) \Big|_{?}^{?}$$



$$\frac{x}{r} = \sin \theta \quad x = r \sin \theta$$

$$\frac{\sqrt{r^2 - x^2}}{r} = \cos \theta \quad dx = r \cos \theta \, d\theta$$

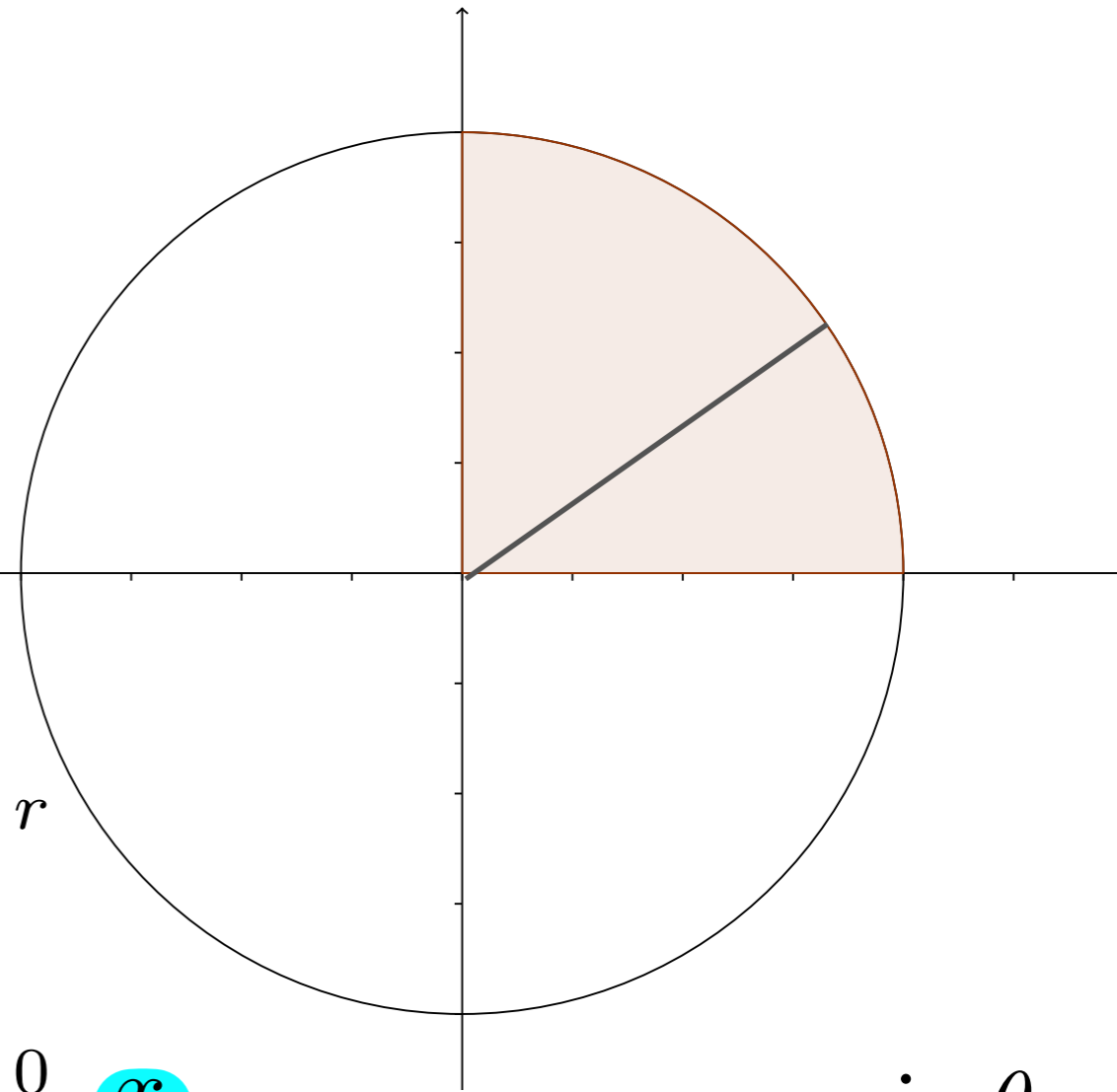
$$\sqrt{r^2 - x^2} = r \cos \theta$$

Exemple

$$\text{Aire}_o = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$= 2r^2 \left(\arcsin \left(\frac{x}{r} \right) + \sin \theta \cos \theta \right) \Big|_0^r$$

$$= 2r^2 \left(\arcsin \left(\frac{x}{r} \right) + \frac{x \sqrt{r^2 - x^2}}{r^2} \right) \Big|_0^r$$

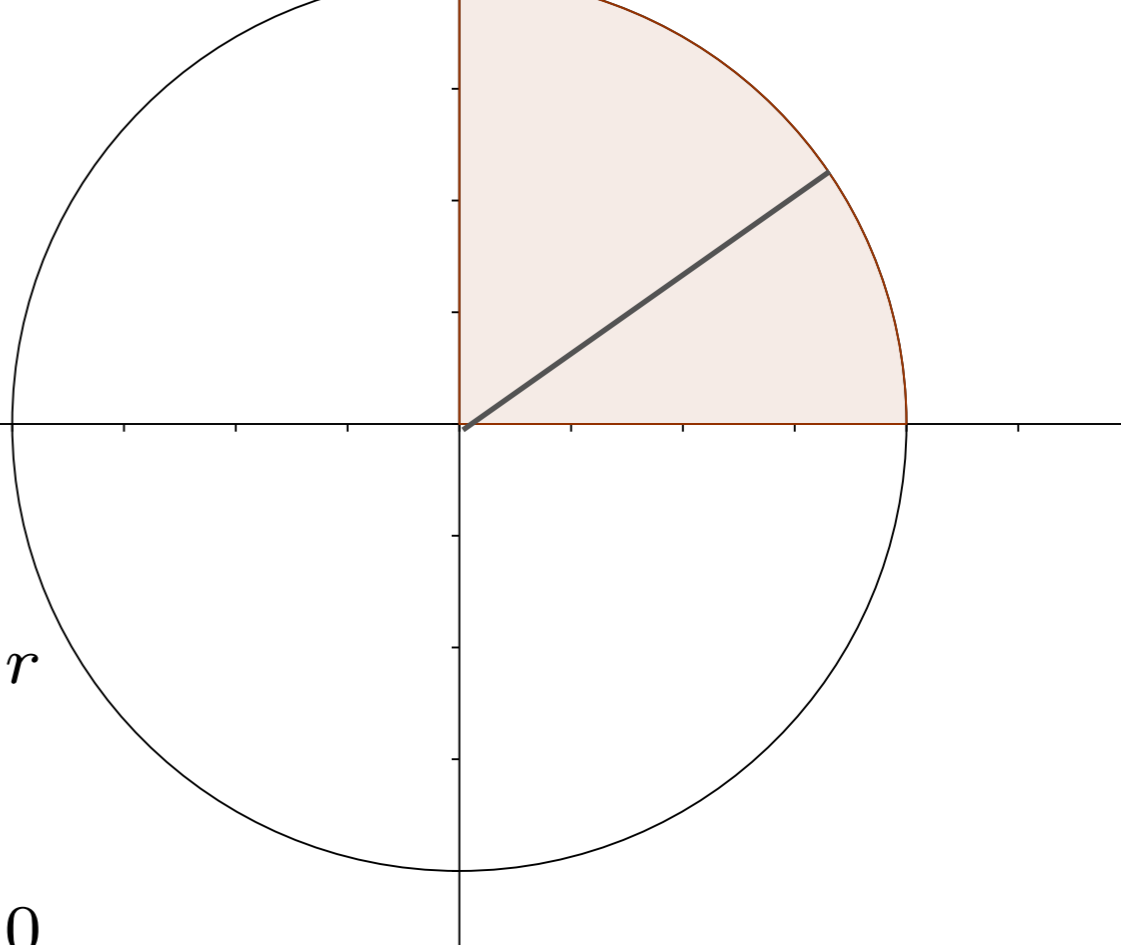


$$\frac{x}{r} = \sin \theta \quad x = r \sin \theta$$

$$\frac{\sqrt{r^2 - x^2}}{r} = \cos \theta \quad dx = r \cos \theta d\theta$$

$$\sqrt{r^2 - x^2} = r \cos \theta$$

Exemple

$$\begin{aligned} \text{Aire}_o &= 4 \int_0^r \sqrt{r^2 - x^2} \, dx \\ &= 2r^2 \left(\arcsin \left(\frac{x}{r} \right) + \sin \theta \cos \theta \right) \Big|_0^r \\ &= 2r^2 \left(\arcsin \left(\frac{x}{r} \right) + \frac{x\sqrt{r^2 - x^2}}{r^2} \right) \Big|_0^r \\ &= 2r^2 \left(\arcsin 1 + \frac{r\sqrt{r^2 - r^2}}{r^2} \right) - 2r^2 \left(\arcsin 0 + \frac{0\sqrt{r^2 - 0^2}}{r^2} \right) \\ &= 2r^2 \left(\frac{\pi}{2} \right) = \pi r^2 \end{aligned}$$


The diagram shows a circle centered at the origin of a Cartesian coordinate system. The circle is divided into four quadrants. The sector in the first quadrant, bounded by the x-axis, the y-axis, and the arc, is shaded in light brown. A line segment from the origin to the arc represents the radius. The angle between the x-axis and this radius is labeled with a question mark. The x-axis and y-axis are marked with tick marks.

Exemple

$$\text{Aire}_o = 4 \int_0^r \sqrt{r^2 - x^2} dx$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$= 4r^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= 2r^2 \left(\theta + \frac{\sin(2\theta)}{2} \right) \Big|_0^{\frac{\pi}{2}}$$

$$= 2r^2 \left(\frac{\pi}{2} + \frac{\sin \pi}{2} \right) - 2r^2 \left(0 + \frac{\sin 0}{2} \right)$$

$$= \pi r^2$$

$$x = r$$

$$\sin \theta = \frac{r}{r} = 1$$

$$\theta = \frac{\pi}{2}$$

$$x = 0$$

$$\sin \theta = \frac{0}{r} = 0$$

$$\theta = 0$$

$$\frac{x}{r} = \sin \theta \quad x = r \sin \theta$$

$$dx = r \cos \theta d\theta$$

$$\frac{\sqrt{r^2 - x^2}}{r} = \cos \theta$$

$$\sqrt{r^2 - x^2} = r \cos \theta$$

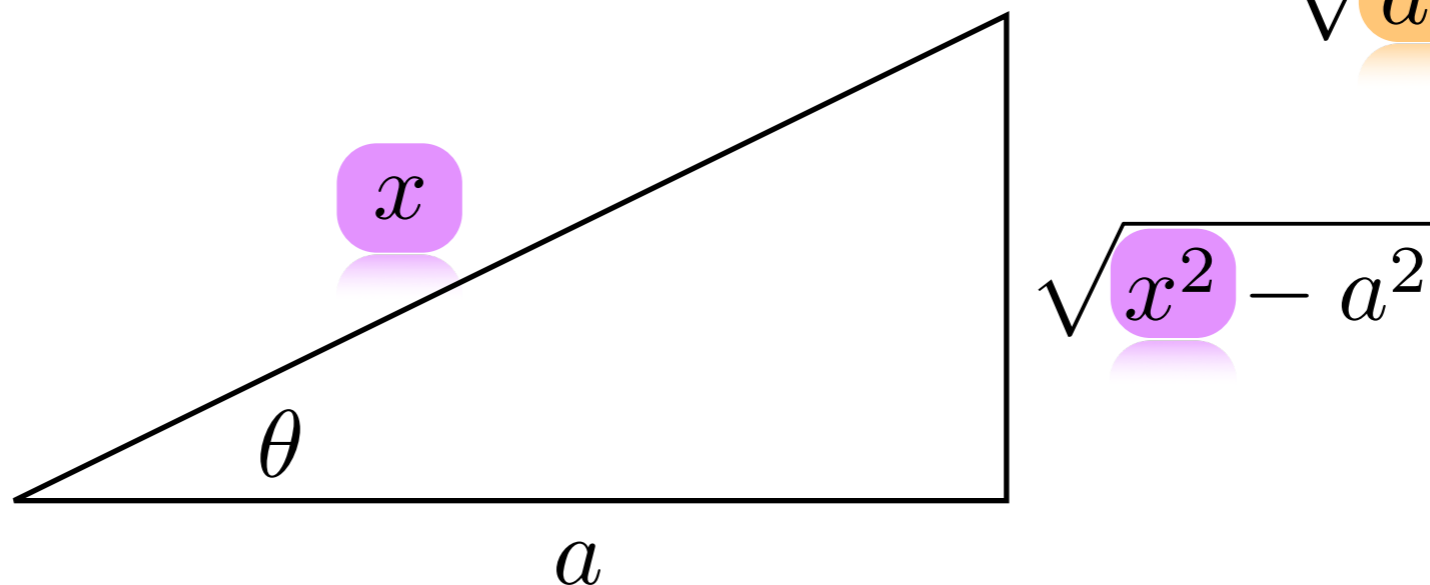
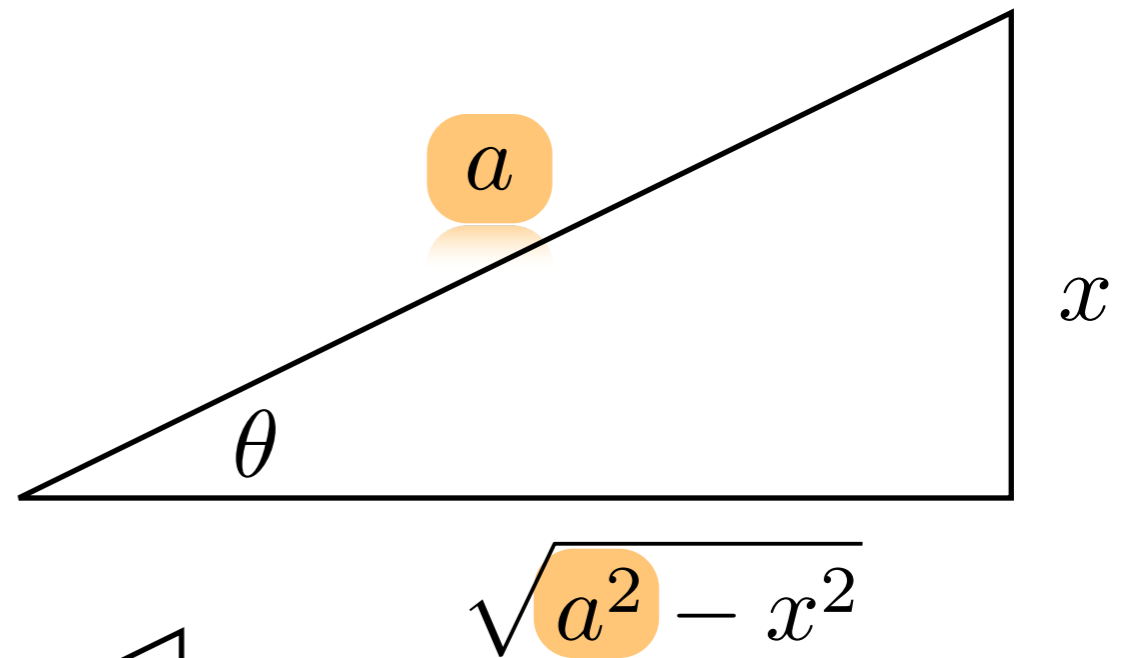
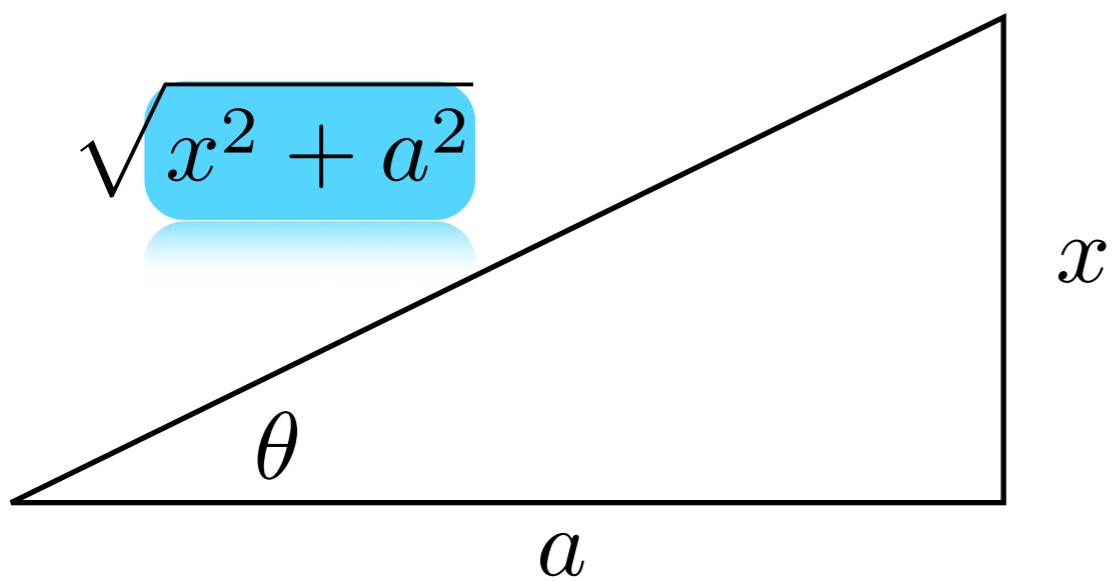
Faites les exercices suivants

Calculer l'aire de l'ellipse définie par l'équation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Aujourd'hui, nous avons vu

✓ Substitution trigonométrique



Devoir:

Section 2.3