Question 1.   (16%)

Évaluer les limites suivantes

a) $\displaystyle\lim_{x\to\frac{\pi}{2}^+}\frac{\tan x}{\ln(x-\frac{\pi}{2})}=\lim_{x\to\frac{\pi}{2}^+}\frac{\sec^2 x}{\frac{1}{x-\frac{\pi}{2}}}=\lim_{x\to\frac{\pi}{2}^+}\frac{x-\frac{\pi}{2}}{\cos^2 x}=\lim_{x\to\frac{\pi}{2}^+}\frac{1}{-2\cos x\sin x}=\frac{1}{0^+}=\infty$

b) $A=\displaystyle\lim_{x\to 0^+}(\sec(5x))^{\csc(3x)}$

$\ln A=\displaystyle\lim_{x\to 0^+}\ln(\sec(5x))^{\csc(3x)}=\lim_{x\to 0^+}\csc(3x)\ln(\sec(5x))=\lim_{x\to 0^+}\frac{\ln(\sec(5x))}{\sin(3x)}=\lim_{x\to 0^+}\frac{\frac{5\sec(5x)\tan(5x)}{\sec(5x)}}{3\cos(3x)}=$

$\displaystyle\lim_{x\to 0^+}\frac{5\sin(5x)}{3\cos(3x)\cos(5x)}=0$ donc, $A=e^{\ln A}=e^0=1$

Question 2.   (32%)

Calculer les intégrales indéfinie suivante.

a) $\displaystyle\int \pi^2-4x^5+\frac{2}{\sqrt[7]{x}}+2\tan x-3^x\,dx=\pi^2 x-\frac{2x^6}{3}+\frac{7x^{\frac{6}{7}}}{3}+2\ln|\sec x|-\frac{3^x}{\ln 3}+C$

b) $\displaystyle\int\frac{\ln(\sqrt{x}\,)}{x}\,dx=\frac{1}{2}\int\frac{\ln x}{x}\,dx,$ $\qquad\qquad\qquad\qquad\qquad u=\ln x,\quad du=\frac{dx}{x}$

$=\displaystyle\int\frac{u}{2}\,du=\frac{u^2}{4}+C=\frac{\ln^2 x}{4}+C$

c) $\displaystyle\int\frac{x^3-x^2+x-7}{x^2+4}\,dx=\int x-1-3\frac{x+1}{x^2+4}\,dx=\frac{x^2}{2}-x-3\int\frac{x}{x^2+4}\,dx-3\int\frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)}\,dx$

$u=x^2+4,\quad du=2x\,dx,\quad v=\frac{x}{2},\quad dv=\frac{dx}{2}$

$=\dfrac{x^2}{2}-x-\dfrac{3}{2}\displaystyle\int\frac{1}{u}\,du-\frac{3}{2}\int\frac{1}{v^2+1}\,dv=\frac{x^2}{2}-x-\frac{3}{2}\ln|u|-\frac{3}{2}\arctan v+C$

$=\dfrac{x^2}{2}-x-\dfrac{3}{2}\ln|x^2+4|-\dfrac{3}{2}\arctan\dfrac{x}{2}+C$

d) $\displaystyle\int\frac{\tan x\sin(\tan x)}{\cos x\sin x}\,dx=\int\frac{\sin(\tan x)}{\cos^2 x}\,dx=\int\sin(\tan x)\sec^2 x\,dx$ $\qquad u=\tan x,\ du=\sec^2 x\,dx$

$=\displaystyle\int\sin u\,du=-\cos u+C=-\cos(\tan x)+C$

Question 3.   (9%)

$$\sum_{k=10}^{20}2k-3=2\sum_{k=10}^{20}k-3\sum_{k=10}^{20}1=2\left(\sum_{k=1}^{20}k-\sum_{k=1}^{9}k\right)-3(20-10+1)$$

$$=2\left(\frac{(20)(21)}{2}-\frac{(9)(10)}{2}\right)-3(11)=2(210-45)-33=297$$

Question 4.   (15%)

Soit la fonction $f(x) = 9 - x^2$

a) $s_4 = \dfrac{1}{2}f\left(\dfrac{3}{2}\right) + \dfrac{1}{2}f\left(\dfrac{4}{2}\right) + \dfrac{1}{2}f\left(\dfrac{5}{2}\right) + \dfrac{1}{2}f\left(\dfrac{6}{2}\right) = \displaystyle\sum_{k=1}^{4}\dfrac{1}{2}\left(9 - \left(1 + \dfrac{k}{2}\right)^2\right)$

b) $s_4 = \dfrac{1}{2}\left(9 - \dfrac{9}{4} + 9 - 4 + 9 - \dfrac{25}{4} + 9 - 9\right) = \dfrac{1}{2}\left(23 - \dfrac{34}{4}\right) = \dfrac{1}{2}\left(\dfrac{92 - 34}{4}\right) = \dfrac{58}{8} = \dfrac{29}{4}$

c) $\displaystyle\int_1^3 9 - x^2 \ dx = 9x - \dfrac{x^3}{3}\Big|_1^3 = (27 - 9) - \left(9 - \dfrac{1}{3}\right) = \dfrac{28}{3}$

Question 5.   (10%)

$x^2 + 3x + 3 = 2x + 9 \Longrightarrow x^2 + x - 6 = 0 \Longrightarrow (x + 3)(x - 2) = 0$

Donc les points d'intersections sont $x = -3$ et $x = 2$.

$A = \displaystyle\int_{-3}^{2} -x^2 - x + 6 \ dx + \int_{2}^{3} x^2 + x - 6 \ dx = -\dfrac{x^3}{3} - \dfrac{x^2}{2} + 6x\Big|_{-3}^{2} + \dfrac{x^3}{3} + \dfrac{x^2}{2} - 6x\Big|_{2}^{3}$

$= \left(-\dfrac{8}{3} - 2 + 12\right) - \left(9 - \dfrac{9}{2} - 18\right) + \left(9 + \dfrac{9}{2} - 18\right) - \left(\dfrac{8}{3} + 2 - 12\right) = \dfrac{71}{3}$

Question 6.   (9%)

a) Vrai car $(x\ln x - x + C)' = \ln x + x\dfrac{1}{x} - 1 + 0 = \ln x$

b) Faux car $f(x) = x$ est toujours croissante mais $\displaystyle\int_{-1}^{0} x \ dx = \dfrac{x^2}{2}\Big|_{-1}^{0} = -\dfrac{1}{2} < 0$

c) Faux car $\dfrac{y}{y^2 - 4}$ a une asymptote en x=2 donc le théorème fondamental ne s'applique pas.