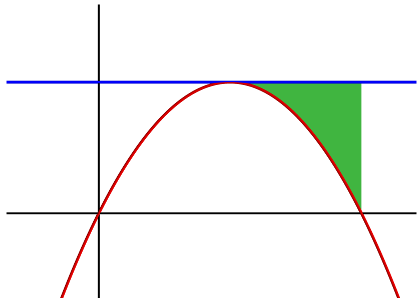
Examen 3 (solutions)

Question 1. (30%)

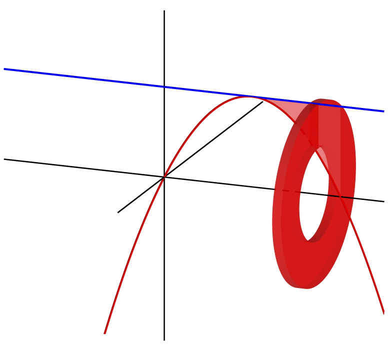


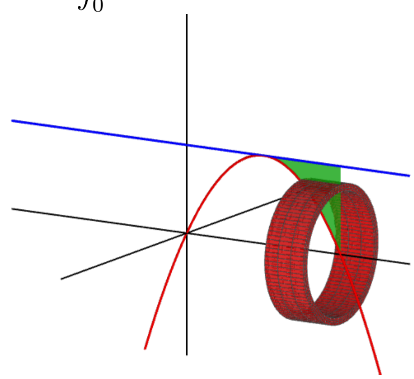
Pour trouver la réciproque de $f(x)$, il suffit de faire une complétion de carré :

$$\begin{aligned} y = -(x^2 - 2x + 1 - 1) &\iff y = 1 - (x-1)^2 \\ &\iff (x-1)^2 = 1 - y \\ &\iff x - 1 = \pm\sqrt{1-y} \\ &\iff x = \pm\sqrt{1-y} + 1 \end{aligned}$$

a) Disques : $\pi \int_1^2 (1)^2 - (-x^2+2x)^2 \, dx$

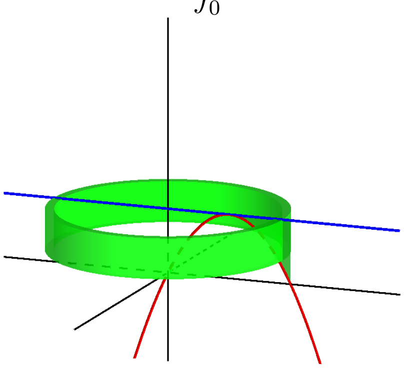
Tubes : $2\pi \int_0^1 y(2 - (\sqrt{1-y}+1)) \, dy$

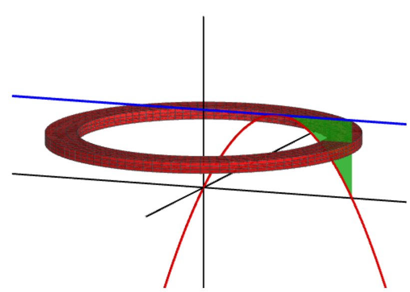




b) Disques : $\pi \int_0^1 (2)^2 - (\sqrt{1-y}+1)^2 \, dy$

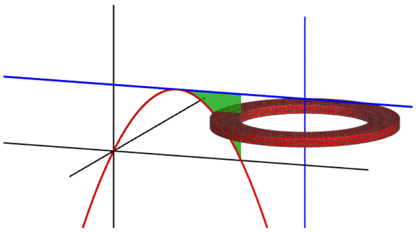
Tubes : $2\pi \int_1^2 x(1 - (-x^2+2x)) \, dx$

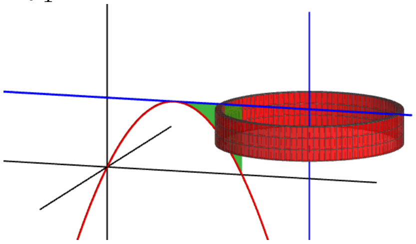




c) Disques : $\pi \int_0^1 (3-\sqrt{1-y}+1)^2 - (3-2)^2 \; dy$

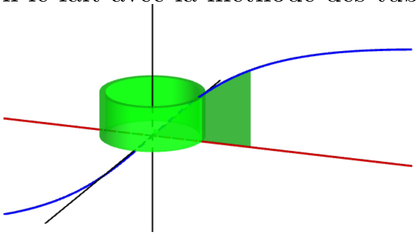
Tubes : $2\pi \int_1^2 (3-x)(1-(-x^2+2x)) \; dx$





Question 2. (10%)

Si on le fait avec la méthode des tubes :



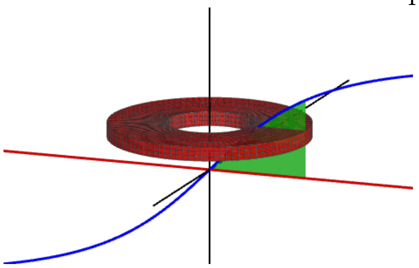
$$2\pi \int_0^1 x \arctan x \; dx = \frac{2\pi x^2}{2} \arctan x \Big|_0^1 - \frac{2\pi}{2} \int_0^1 \frac{x^2}{x^2+1} \; dx \qquad u = \arctan x \quad dv = x \; dx$$

$$= \pi \frac{\pi}{4} - 0 - \pi \int_0^1 1 - \frac{1}{x^2+1} \; dx \qquad du = \frac{dx}{x^2+1} \quad v = \frac{x^2}{2}$$

$$= \frac{\pi^2}{4} - \pi(x - \arctan x)\Big|_0^1$$

$$= \frac{\pi^2}{4} - \pi(1 - \frac{\pi}{4}) = \frac{\pi^2}{2} - \pi$$

Si on le fait avec la méthode des disques :



$$\pi \int_0^{\frac{\pi}{4}} 1^2 - \tan^2 y \; dy = \pi \int_0^{\frac{\pi}{4}} 1 - (\sec^2 y - 1) \; dy$$

$$= \pi(2y - \tan y)\Big|_0^{\frac{\pi}{4}}$$

$$= \pi\left(\frac{2\pi}{4} - 1\right) = \frac{\pi^2}{2} - \pi$$

Question 3. (15%)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sqrt{1+(\ln(\sec x)')^2}\ dx &= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\ dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\ dx = \int_0^{\frac{\pi}{4}} \sec x\ dx \\ &= \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}} = \ln\left|\frac{2}{\sqrt{2}}+1\right| - \ln|1+0| \\ &= \ln|\sqrt{2}+1|\end{aligned}$$

Question 4. (10%)

$$\begin{aligned}2\pi\int_0^{\sqrt{2}} x\sqrt{1+((x^2)')^2}\ dx = 2\pi\int_0^{\sqrt{2}} x\sqrt{1+4x^2}\ dx &= \frac{2\pi}{8}\int_?^? \sqrt{u}\ du \\ &= \frac{\pi}{4}\frac{2\sqrt{(1+4x^2)^3}}{3}\Bigg|_0^{\sqrt{2}} = \frac{\pi}{6}(27-1) = \frac{13\pi}{3}\end{aligned}$$

Question 5. (20%)

a)

$$\int_0^2 \frac{dx}{x-2} = \lim_{t\to 2^-}\int_0^t \frac{dx}{x-2} = \lim_{t\to 2^-} \ln|x-2|\Big|_0^t = \lim_{t\to 2^-} \ln|t-2| - \ln|-2| = -\infty$$

Donc diverge

b)

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{N\to-\infty}\int_N^0 \frac{dx}{1+x^2} + \lim_{M\to\infty}\int_0^M \frac{dx}{1+x^2} = \lim_{N\to-\infty} \arctan x\Big|_N^0 + \lim_{M\to\infty} \arctan x\Big|_0^M \\ &= \left(0-\frac{-\pi}{2}\right)+\left(\frac{\pi}{2}-0\right) = \pi\end{aligned}$$

Donc converge

Question 6. (15%)

$$\begin{aligned}\frac{y'}{e^x} = \sec y &\iff \frac{dy}{\sec y} = e^x\ dx \iff \cos y\ dy = e^x\ dx \\ &\iff \int \cos y\ dy = \int e^x\ dx \\ &\iff \sin y = e^x + C\end{aligned}$$

Pour trouver C :

$$\sin\frac{\pi}{6} = e^2 + C \Longrightarrow C = \frac{1}{2} - e^2$$

Donc

$$y = \arcsin\left(e^x + \frac{1}{2} - e^2\right)$$